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ADS 3-MANIFOLDS AND HIGGS BUNDLES

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Abstract. In this paper we investigate the relationships between closed AdS 3-manifolds and Higgs bundles. We have a new way to construct AdS structures that allows us to see many of their properties explicitly, for example we can recover the very recent formula by Tholozan for their volume.

We give natural foliations of the AdS structure with time-like geodesic circles and we use these circles to construct equivariant minimal immersions of the Poincaré disc into the Grassmannian of time-like 2-planes of $\mathbb{R}^{2,2}$.

1. Introduction

The non-abelian Hodge correspondence, mainly developed by Hitchin [12], Donaldson [6], Simpson [19] and Corlette [4], gives a homeomorphism between the moduli space of polystable G-Higgs bundles for a semi-simple Lie group G over a Riemann surface Σ, and the character variety of representations of $\pi_1(\Sigma)$ into G. Because of this correspondence, the theory of Higgs bundles has been very useful in the study of the topological structure of the character varieties. More difficult is to get geometric information about a single representation from the corresponding Higgs bundle. This problem is one of the motivations for this paper.

In this paper we investigate the relationships between Higgs bundles and geometric structures on 3-manifolds. These relationships were explored in Baraglia's thesis [1], and more recently in the present work, in some other work in progress of the authors and of the authors with Brian Collier. A geometric structure is determined by a developing pair consisting of a representation and a developing map (see Section 2). The Higgs bundle encodes the representation through the non-abelian Hodge correspondence. To obtain a geometric structure, we need to describe the developing map in terms of the Higgs bundle data.

The geometric structures we study in this paper are the 3-dimensional Anti-de Sitter structures (abbreviated as AdS structures), Lorentz metrics with constant curvature -1. They initially arose as models for general relativity, but their rich mathematical theory makes them very interesting geometric objects, independently

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on physical applications. The theories of closed and open AdS 3-manifolds are quite different, here we will deal only with closed AdS 3-manifolds.

The AdS 3-manifolds are locally modeled on the Anti-de Sitter 3-space $\mathbb{M}$ (see Section 2), whose isometry group is $O(2,2)$. If ρ_1, ρ_2 are two representations of the fundamental group of a closed surface S in $SL(2,\mathbb{R})$, then their tensor product $\rho_1 \otimes \rho_2$ is a representation in $O(2,2)$. We will use the theory of Higgs bundles for $SL(2,\mathbb{R})$ to describe the Higgs bundles for the tensor product $\rho_1 \otimes \rho_2$, and to construct a circle bundle U over S. To obtain our results, we also complete the description of the Higgs bundles for $SL(2,\mathbb{R})$; see Theorems 3.1 and 4.1.

If ρ_1, ρ_2 satisfy a condition of domination depending on the choice of a conformal structure, using the technique of the graph of geometric structures, we construct an AdS structure on U with holonomy $\rho_1 \otimes \rho_2$ (see Section 5).

In Section 6, using some results by Deroin-Tholozan [5] and Tholozan [21], one can see that the domination condition we consider is equivalent to the classical domination condition defined by Salein. Therefore we give a new proof of a fundamental theorem originally proven by Salein [18], and later reproven in [10].

In our proof, the structure we construct has a natural parametrization and some properties of the AdS manifold can be seen easily using this parametrization. We can determine the underlying topology of the AdS manifold, and we can see that the fibers of the circle bundle U are time-like geodesics for the AdS structure (see Section 7).

We can also compute the volume of the AdS structure with an explicit formula that shows that the volume only depends on the Euler numbers of the representations ρ_1, ρ_2. This result was cited as Question 2.3 in the list of open problems [2]. It was first answered some months before us in Tholozan's thesis [22]; see also [23]. Labourie [17] then related the volumes of AdS 3-manifolds with the Chern-Simons invariants. In Section 8, we give a new proof of the formula.

In this way, we can recover much of the theory of closed AdS 3-manifolds, with new methods based mainly on Higgs bundles, harmonic maps and the study of solutions of Hitchin's equations.

In Section 7, given an AdS manifold with holonomy $\rho_1 \otimes \rho_2$, we use the AdS structure on the fibers of the circle bundle to construct a $\rho_1 \otimes \rho_2$-equivariant minimal immersion of the hyperbolic plane in the Grassmannian $\mathrm{Gr}^+(2,4)$ of time-like 2-planes in $\mathbb{R}^{2,2}$, equipped with its natural pseudo-Riemannian metric. This is an original result for this paper. This gives a characterization of the representations that admit equivariant minimal surfaces in $\mathrm{Gr}^+(2,4)$. We can also characterize the conformal structure of the minimal surface: it is the conformal structure with vanishing Pfaffian found by Tholozan in [21].

At first glance, it seems that we just construct and study some examples of closed AdS 3-manifolds. We would like to remark that these examples lead to the general case, by understanding them we can understand all closed AdS 3-manifolds: from the results of Klingler [15], Kulkarni and Raymond [16] and Kassel [14], given a closed AdS 3-manifold Y, there exist two representations ρ_1, ρ_2 of a surface group in $SL(2,\mathbb{R})$ with the property that ρ_1 strictly dominates ρ_2, such that Y and the quotient $\mathbb{M}/\rho_1 \otimes \rho_2$ have a common finite covering.

2. Anti-de Sitter 3-manifolds

Given a 3-manifold N, an *Anti-de Sitter structure* on N, or briefly an *AdS structure* on N, is a Lorentz metric of constant curvature -1. Recall that a *Lorentz metric* is a pseudo-Riemannian metric of signature $(2,1)$.

Every AdS structure is locally isometric to a model space, the *Anti-de Sitter space*, that can be described explicitly as follows. Let Q denote a non-degenerate symmetric bilinear form on $\mathbb{R}^4$ of signature $(2,2)$. Sometimes, we write the pair $(\mathbb{R}^4, Q)$ as $\mathbb{R}^{2,2}$. The *Anti-de Sitter space* is identified with the unit sphere of Q:

$$\mathbb{M} = \{x \in \mathbb{R}^4 \mid Q(x,x) = 1\}.$$

For every point $x \in \mathbb{M}$, the tangent space to $\mathbb{M}$ at x is given by $x^{\perp Q}$, the orthogonal with reference to Q. The Lorentz metric is given by the restriction of the bilinear form Q to the tangent space $T_x\mathbb{M} = x^{\perp Q}$, a bilinear form of signature $(2,1)$. A tangent vector $v \in T_x\mathbb{M}$ is *time-like* if $Q(v,v) > 0$, and *space-like* if $Q(v,v) < 0$.

The group of isometries of $\mathbb{M}$ is the orthogonal group preserving Q, here denoted by $O(2,2)$. Here we will mainly use the subgroup $SO_0(2,2)$, the group of isometries preserving both space and time orientation. It can be identified with $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$ in the following way. Let ω be a volume form on $\mathbb{R}^2$. Given $A, B \in SL(2,\mathbb{R})$, they act on $\mathbb{R}^2$ preserving ω, hence $A \otimes B$ acts on $\mathbb{R}^2 \otimes \mathbb{R}^2$ preserving the bilinear form $Q = \omega \otimes \omega$. If $A = B = -\mathrm{Id}$, the action is trivial. Note that Q is symmetric and of signature $(2,2)$.

Given a 3-manifold N, we want to construct AdS structures on N. We will use the model space $\mathbb{M}$ and mirror its local geometry on N. More precisely, N is covered by open charts $\{(U_i, \phi_i)\}_{i\in I}$, where every ϕ_i is a diffeomorphism from U_i to an open subset of $\mathbb{M}$. Whenever two open sets U_i, U_j intersect, the transition functions have to be locally the restrictions of elements of $O(2,2)$. If the transition functions lie in $SO_0(2,2)$, the structure has space and time orientation.

This construction comes from the theory of geometric structures, or (G,X)-structures, where G is a Lie group acting transitively and effectively on a manifold X. In our case, $G = SO_0(2,2)$, and $X = \mathbb{M}$. For the general theory of geometric structures, the reader is referred to [7] or [24]. A (G,X)-structure is defined as a maximal atlas of charts $\{(U_i, \phi_i)\}_{i\in I}$ satisfying the properties given above.

An equivalent way to see a geometric structure is via the *developing pair*, (D, ρ), where $\rho : \pi_1(N) \to G$ is a representation called *holonomy representation*, and $D : \widetilde{N} \to X$ is a ρ-equivariant local diffeomorphism, called a *developing map*. This paper was inspired by the following question: given a representation $\rho : \pi_1(N) \to G$, how can we construct a geometric structure on N with holonomy ρ?

To address this question, we can use another equivalent way to see a geometric structure, called the *graph of a geometric structure* (see also [7] for more details). Given a representation $\rho : \pi_1(N) \to G$, and an effective action of G on X, we can construct a flat bundle $p : B \to N$ over N with fiber X, structure group G, and monodromy equal to ρ. The total space is given by $B = X_\rho = \left(\widetilde{N} \times X\right)/\pi_1(N)$, where the group $\pi_1(N)$ acts diagonally on the product. The space X_ρ has a natural map to $N = \widetilde{N}/\pi_1(N)$ which is a fiber bundle. The flat structure on the product $\widetilde{N} \times X$ descends to the quotient. The monodromy of the flat bundle $B = X_\rho$ is exactly the representation ρ, so the flat bundle X_ρ encodes the representation.

To construct a geometric structure with holonomy ρ, we only need to add one more piece of information, the developing map. We need to interpret the developing map in the language of flat bundles. The notion of a ρ-equivariant map can be translated very naturally: it is a section of the bundle X_ρ. Given a section $s : N \to X_\rho$, it can be lifted to the universal covering $\widetilde{s} : \widetilde{N} \to \widetilde{X_\rho}$, where $\widetilde{X_\rho}$ is the pull back of X_ρ to $\widetilde{N}$. Since $\widetilde{X_\rho} = \widetilde{N} \times X$, the projection on the second component gives a ρ-equivariant map $\widetilde{N} \to X$. Following [7], we call *transverse section* a section whose associated ρ-equivariant map is a local diffeomorphism, i.e., a developing map of a geometric structure.

In this paper we use the technique of the graph of geometric structures to construct AdS structures with prescribed holonomy. Let S be an orientable closed surface of genus $g \geq 2$, and $\rho_1, \rho_2 : \pi_1(S) \to SL(2, \mathbb{R})$ be two reductive representations. Their tensor product $\rho = \rho_1 \otimes \rho_2$ is a representation in $SO_0(2, 2)$ by the isomorphism $SO_0(2, 2) = (SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\mathbb{Z}_2$ as explained in Section 2. Let $p : U \to S$ be a circle bundle over S. The representation ρ induces

$$\bar{\rho} = \rho \circ p_* : \pi_1(U) \to SO_0(2, 2),$$

a representation of $\pi_1(U)$ that is trivial on the fibers. We aim to construct an AdS structure on the 3-manifold U with holonomy $\bar{\rho}$.

We denote by E the vector bundle $(\mathbb{R}^4)_\rho \to S$. Since ρ is a representation in $SO_0(2, 2)$, the bundle E is naturally equipped with a symmetric bilinear form Q of signature $(2, 2)$ on every fiber. The fiber bundle $\mathbb{M}_\rho \to S$ is a sub-bundle of the bundle E, whose fibers are defined by the equation $Q(v, v) = 1$. To construct an AdS structure on U, we need to consider the pull-back bundles $p^*E \to U$ and $p^*\mathbb{M}_\rho = \mathbb{M}_{\bar{\rho}} \to U$, and find a transverse section $s : U \to \mathbb{M}_{\bar{\rho}}$.

It is not possible to achieve this for every representation ρ_1, ρ_2 of $\pi_1(S)$, and for every circle bundle U, so we need to add some hypotheses. To choose a suitable circle bundle U, we need to understand the structure of the bundle $E \to S$. Denote by $e_1, e_2 \in [2 - 2g, 2g - 2] \cap 2\mathbb{Z}$ the Euler numbers of ρ_1, ρ_2 respectively. We show in the next section that the bundle E can be split as a direct sum of two vector bundles of rank 2, $E = F_1 \oplus F_2$ such that F_1 has Euler class $|e_2 - e_1|$ and F_2 has Euler class $|e_1 + e_2|$. The two sub-bundles can be chosen such that $F_1 \perp_Q F_2$, and such that F_1 is time-like and F_2 is space-like.

Let us choose U as the circle bundle $U = \{v \in F_1 \mid Q(v, v) = 1\}$. This is the unit part of F_1, hence a circle bundle with Euler class $|e_2 - e_1|$. It has a natural section $s : U \to \mathbb{M}_{\bar{\rho}}$, the tautological section that associates to every point v of U the same point v seen as a point of $\mathbb{M}_{\bar{\rho}}$.

The last thing needed to construct the AdS structure on U is to verify the transversality condition of the section s. Since E is a vector bundle, its flat structure is described by a flat connection ∇, and the transversality condition can be formulated in terms of the covariant derivatives of s with respect to ∇.

We prove the above claims in Sections 3 and 5, using the tool given by Higgs bundles.

3. Higgs bundles and flat bundles

The two reductive $SL(2, \mathbb{R})$ representations ρ_1, ρ_2 of the previous section, with Euler numbers e_1, e_2, correspond to flat unimodular vector bundles $(E_1, \nabla_1, \omega_1)$,

$(E_2, \nabla_2, \omega_2)$, where the ∇_i's are flat connections and the ω_i's are ∇_i-parallel volume forms. In a local frame, let us write $\nabla_i = d + A_i$, where A_i is the connection form.

The representation $\rho = \rho_1 \otimes \rho_2$ corresponds to the flat bundle (E, ∇, Q), where $E = E_1 \otimes E_2$, $Q = \omega_1 \otimes \omega_2$, and the connection can be described, in the compatible local tensor frame, as $\nabla = d + A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$.

We will also consider the corresponding complexified bundles $(E_i^{\mathbb{C}}, \nabla_i^{\mathbb{C}}, \omega_i^{\mathbb{C}})$ and $(E^{\mathbb{C}}, \nabla^{\mathbb{C}}, Q^{\mathbb{C}})$. There are anti-linear involutions $\tau_i : E_i^{\mathbb{C}} \to E_i^{\mathbb{C}}$ and $\tau : E^{\mathbb{C}} \to E^{\mathbb{C}}$, such that $E_i = \{v \in E_i^{\mathbb{C}} \mid \tau_i(v) = v\}$ and $E = \{v \in E^{\mathbb{C}} \mid \tau(v) = v\}$. The involution is called the *real structure*. Hence the full structure on the complexified bundles is $(E_i^{\mathbb{C}}, \nabla_i^{\mathbb{C}}, \omega_i^{\mathbb{C}}, \tau_i)$ and $(E^{\mathbb{C}}, \nabla^{\mathbb{C}}, Q^{\mathbb{C}}, \tau)$.

To describe more explicitly the bundles $E_i^{\mathbb{C}}, E^{\mathbb{C}}$, we will use the theory of Higgs bundles, initially introduced by Hitchin [12] and Donaldson [6]. Choose a complex structure Σ on the surface S and denote by K its canonical bundle. There is a unique ρ_i-equivariant harmonic map ψ_i from $\widetilde{\Sigma}$ to $SL(2, \mathbb{C})/SU(2)$, the space of all Hermitian metrics on $\mathbb{C}^2$. Hence, the ρ_i-equivariant harmonic map ψ_i corresponds to a harmonic Hermitian metric H_i on the vector bundle $E_i^{\mathbb{C}}$.

Using this Hermitian metric H_i, we can uniquely decompose the flat connection $\nabla_i^{\mathbb{C}}$ into a sum of a unitary connection ∇^{H_i} and a 1-form ψ_i valued in the skew-Hermitian endomorphism of the bundle $E_i^{\mathbb{C}}$. The $(0,1)$-part of ∇^{H_i} defines a holomorphic structure on $E_i^{\mathbb{C}}$, and the $(1,0)$-part of ψ_i gives a holomorphic Higgs field $\phi_i \in H^0(\Sigma, \mathrm{End}(E_i^{\mathbb{C}}) \otimes K)$. Note that the harmonicity of the Hermitian metric H_i ensures the holomorphicity of ϕ_i. In this way, from a flat $SL(2, \mathbb{C})$ connection we obtain a Higgs bundle for $SL(2, \mathbb{C})$. The flatness of the connection ∇_i is equivalent to the condition that H_i is the solution to Hitchin's equation

$$F_{\nabla^{H_i}} + [\phi_i, \phi_i^{*H_i}] = 0, \tag{3.1}$$

where $F_{\nabla^{H_i}}$ is the curvature of ∇^{H_i} and $\phi_i^{*H_i}$ is the adjoint of ϕ_i with respect to H_i. Conversely, given a stable Higgs bundle, there exists a unique Hermitian metric solving Hitchin's equation, giving rise to a flat connection $\nabla_i = \nabla^{H_i} + \phi_i + \phi_i^{*H_i}$. This is the non-abelian Hodge correspondence for $SL(2, \mathbb{C})$. It was later generalized to reductive Lie groups by Corlette [4] and Simpson [19].

In our case, the flat connection on $E_i^{\mathbb{C}}$ was actually an $SL(2, \mathbb{R})$-connection and the associated ρ_i-equivariant harmonic maps ψ_i lie in a totally geodesic submanifold $\mathbb{H} = SL(2, \mathbb{R})/SO(2) \subset SL(2, \mathbb{C})/SU(2)$. To parametrize the $SL(2, \mathbb{R})$-representations, we need to impose some extra structure on the Higgs bundles, which we call Higgs bundles for $SL(2, \mathbb{R})$. This is described by Hitchin in [12], and a more explicit description can be found in Gothen's thesis [9], Section 2.2.2, where he deals with the more general case of Higgs bundles for $Sp(2n, \mathbb{R})$. From this description one can see that the holomorphic bundle $E_i^{\mathbb{C}}$ splits as a direct sum of a holomorphic line bundle and its inverse:

$$E_1^{\mathbb{C}} = L \oplus L^{-1}, \qquad E_2^{\mathbb{C}} = N \oplus N^{-1}, \tag{3.2}$$

where $\deg L = \frac{1}{2}e_1$ and $\deg N = \frac{1}{2}e_2$ (recall that e_1, e_2 are even integers).

In terms of this splitting, the Higgs fields are of the following form:

$$\phi_1 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}, \tag{3.3}$$

with $\alpha \in H^0(\Sigma, L^2K)$, $\beta \in H^0(\Sigma, L^{-2}K)$, $\gamma \in H^0(\Sigma, N^2K)$ and $\delta \in H^0(\Sigma, N^{-2}K)$.

Theorem 3.1. *In terms of the splitting of E_i as a direct sum of two line bundles, the metric H_i is diagonal, i.e., it can be written as*

$$H_1 = \begin{pmatrix} k^{-1} & 0 \\ 0 & k \end{pmatrix}, \qquad H_2 = \begin{pmatrix} h^{-1} & 0 \\ 0 & h \end{pmatrix},$$

where $k \in \Gamma(\Sigma, \bar{L} \otimes L)$ and $h \in \Gamma(\Sigma, \bar{N} \otimes N)$.

Proof. Higgs bundles for $SL(2,\mathbb{R})$ have an $SO(2,\mathbb{C})$ structure coming from the pairing between L (or N) and L^{-1} (or N^{-1}), given by $B_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. View $B_i : E_i \to E_i^*$ as a holomorphic isomorphism. Then $B_i^{-1}\phi_i^T B_i$ is again a well-defined Higgs field in $H^0(\mathrm{End}(E_i) \otimes K)$. One can formally check that $\overline{B_i}^T (H_i^T)^{-1} B_i$ is the solution of Hitchin's equation for the Higgs bundle $(E_i, B_i^{-1}\phi_i^T B_i)$. By formula (3.3), $B_i^{-1}\phi_i^T B_i = \phi_i$. Therefore, by the uniqueness of the solution to Hitchin's equation, we have $\overline{B_i}^T (H_i^T)^{-1} B_i = H_i$ (this is essentially Formula (7.2) in Hitchin's [13]). This equation and the condition $\det(H_i) = 1$ imply the statement. □

Choosing a local holomorphic frame $\{e_i, e_i^*\}$, the flat $SL(2,\mathbb{R})$-connection $\nabla_i^{\mathbb{C}}$ is given explicitly by

$$\nabla_i^{\mathbb{C}} = d + A_i = d + H_i^{-1}\partial H_i + \phi_i + \phi_i^{*H_i}.$$

The real structure $\tau_i : E_i^{\mathbb{C}} \to E_i^{\mathbb{C}}$ is given explicitly by:

$$\tau_1 : \quad E_1^{\mathbb{C}} \ni v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} 0 & k \\ k^{-1} & 0 \end{pmatrix} \bar{v} = \begin{pmatrix} k\bar{v}_2 \\ k^{-1}\bar{v}_1 \end{pmatrix} \in E_1^{\mathbb{C}},$$

$$\tau_2 : \quad E_2^{\mathbb{C}} \ni v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} 0 & h \\ h^{-1} & 0 \end{pmatrix} \bar{v} = \begin{pmatrix} h\bar{v}_2 \\ h^{-1}\bar{v}_1 \end{pmatrix} \in E_2^{\mathbb{C}}.$$

Finally, the volume form $\omega_i^{\mathbb{C}}$ is given by $\omega_i^{\mathbb{C}} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This completes the description of the bundle $(E_i^{\mathbb{C}}, \nabla_i^{\mathbb{C}}, \omega_i^{\mathbb{C}}, \tau_i)$ given by the choice of Σ.

The non-abelian Hodge correspondence respects tensor products, i.e., the Higgs bundle corresponding to the tensor product of two representations is the tensor product of two Higgs bundles; see Simpson [20]. Therefore, to describe the bundle $(E^{\mathbb{C}}, \nabla^{\mathbb{C}}, Q^{\mathbb{C}}, \tau)$ we just need to consider the tensor product of Higgs bundles:

- The holomorphic vector bundle is

$$E^{\mathbb{C}} = E_1^{\mathbb{C}} \otimes E_2^{\mathbb{C}} = LN \oplus LN^{-1} \oplus L^{-1}N \oplus L^{-1}N^{-1}.$$

- The Higgs field is given by (see [20])

$$\Phi = \phi_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \phi_2 = \begin{pmatrix} 0 & \gamma & \alpha & 0 \\ \delta & 0 & 0 & \alpha \\ \beta & 0 & 0 & \gamma \\ 0 & \beta & \delta & 0 \end{pmatrix}.$$

The characteristic polynomial of Φ is $P(\Phi) = x^4 - 2(\alpha\beta + \gamma\delta)x^2 + (\alpha\beta - \gamma\delta)^2$. The second coefficient $-2(\alpha\beta + \gamma\delta)$ can be interpreted as the Hopf differential of the associated harmonic map. The square root of the fourth coefficient $\alpha\beta - \gamma\delta$ is usually called the *Pfaffian*. It will be important in Sections 4 and 7.

- The Hermitian metric solution of Hitchin's equation (3.1) is the tensor product

$$H = H_1 \otimes H_2 = \mathrm{diag}(\ h^{-1}k^{-1}, hk^{-1}, h^{-1}k, hk\ ).$$

• With respect to the local holomorphic frame $\{e_1 \otimes e_2, e_1 \otimes e_2^*, e_1^* \otimes e_2, e_1^* \otimes e_2^*\}$, the flat $SO_0(2,2)$ connection is given by $\nabla^{\mathbb{C}} = d + H^{-1}\partial H + \Phi + \Phi^{*_H}$. From now on, all the computations are given with reference to this frame.

The covariant derivatives of $\nabla^{\mathbb{C}}$ with respect to the directions $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are:

$$\begin{aligned}
\nabla^{\mathbb{C}}_{\frac{\partial}{\partial z}} &= \partial + H^{-1}\partial H + \Phi \\
&= \partial + \begin{pmatrix} \partial\log(h^{-1}k^{-1}) & \gamma & \alpha & 0 \\ \delta & \partial\log(hk^{-1}) & 0 & \alpha \\ \beta & 0 & \partial\log(h^{-1}k) & \gamma \\ 0 & \beta & \delta & \partial\log(hk) \end{pmatrix}, \\
\nabla^{\mathbb{C}}_{\frac{\partial}{\partial \bar{z}}} &= \bar{\partial} + \Phi^{*_H} = \bar{\partial} + \begin{pmatrix} 0 & h^2\bar{\delta} & k^2\bar{\beta} & 0 \\ h^{-2}\bar{\gamma} & 0 & 0 & k^2\bar{\beta} \\ k^{-2}\bar{\alpha} & 0 & 0 & h^2\bar{\delta} \\ 0 & k^{-2}\bar{\alpha} & h^{-2}\bar{\gamma} & 0 \end{pmatrix}.
\end{aligned}$$

. • The bilinear form and the real structure are given by:

$$Q^{\mathbb{C}} = \omega_1 \otimes \omega_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\tau : E^{\mathbb{C}} \ni \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & hk \\ 0 & 0 & h^{-1}k & 0 \\ 0 & hk^{-1} & 0 & 0 \\ h^{-1}k^{-1} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \\ \bar{v}_4 \end{pmatrix} = \begin{pmatrix} hk\bar{v}_4 \\ h^{-1}k\bar{v}_3 \\ hk^{-1}\bar{v}_2 \\ h^{-1}k^{-1}\bar{v}_1 \end{pmatrix} \in E^{\mathbb{C}}.$$

The real structure τ leaves invariant the two sub-bundles $LN \oplus L^{-1}N^{-1}$ and $LN^{-1} \oplus L^{-1}N$. Denote by F_1 the real part of $LN^{-1} \oplus L^{-1}N$, and by F_2 the real part of $LN \oplus L^{-1}N^{-1}$, which are the real sub-bundles mentioned in Section 2.

Proposition 3.2. *The vector bundle E splits as $E = F_1 \oplus F_2$, a direct sum of two rank 2 sub-bundles, where F_1 has Euler class $|e_1 - e_2|$ and F_2 has Euler class $|e_1 + e_2|$. The direct sum is Q-orthogonal, F_1 is time-like and F_2 is space-like.*

We should notice that the sub-bundles F_1, F_2 are orientable, but they don't come with a natural orientation. For this reason, the sign of their Euler number is not well defined, only the absolute value is well defined.

Proof. We can write an isomorphism between LN and F_2 as real vector bundles:

$$LN \ni v \to v + \tau v \in F_2.$$

Hence the Euler class of F_2 is the same as the Euler class of LN, a complex line bundle of degree $(e_1+e_2)/2$. Similar argument works for F_1. The rest follows from an easy computation, since Q and τ are both explicit. □

Now we can finally define the circle bundle U mentioned in the previous section:

$$U = \{v \in F_1 \mid Q(v,v) = 1\}.$$

4. The pull-back metrics

In this section we need to analyze more deeply the structure of the ρ_i-equivariant harmonic map ψ_i from $\widetilde{\Sigma}$ to the hyperbolic plane $\mathbb{H}$. Denote by $g_{\mathbb{H}}$ the hyperbolic metric on $\mathbb{H}$, with constant curvature -1, and by $\mathrm{Vol}(g_{\mathbb{H}})$ its volume form. The 2-tensors $\psi_i^* g_{\mathbb{H}}$ and $\psi_i^* \mathrm{Vol}(g_{\mathbb{H}})$ are $\pi_1(\Sigma)$-invariant, hence they descend to 2-tensors on Σ that we will denote by g_i and $\mathrm{Vol}(g_i)$. We will call g_i the *pull-back metric*, even if it is only a symmetric positive semi-definite 2-tensor that can be degenerate at some points.

We will call the anti-symmetric 2-tensor $\mathrm{Vol}(g_i)$ the *pull-back volume form*, even if it is not in general a volume form, because it can have zeros and changes of sign. The Euler number e_i of the representation ρ_i is given by (see [3, sec. 3.5]):

$$e_i = \frac{1}{2\pi}\int_\Sigma \mathrm{Vol}(g_i). \tag{4.1}$$

Now we express g_i and $\mathrm{Vol}(g_i)$ in terms of the Higgs bundle data. The following theorem is new only for non-Fuchsian representations. For Fuchsian representations in $SL(2,\mathbb{R})$, see [12] and [26].

Theorem 4.1. *(1) The pull-back metric g_i is given by*

$$g_1 = 4\alpha\beta dz^2 + 4(k^2|\beta|^2 + k^{-2}|\alpha|^2)dzd\bar{z} + 4\bar{\alpha}\bar{\beta}d\bar{z}^2,$$
$$g_2 = 4\gamma\delta dz^2 + 4(h^2|\delta|^2 + h^{-2}|\gamma|^2)dzd\bar{z} + 4\bar{\gamma}\bar{\delta}d\bar{z}^2.$$

(2) The pull-back volume form $\mathrm{Vol}(g_i)$ is given by

$$\mathrm{Vol}(g_1) = 4(k^{-2}|\alpha|^2 - k^2|\beta|^2)dx\wedge dy,$$
$$\mathrm{Vol}(g_2) = 4(h^{-2}|\gamma|^2 - h^2|\delta|^2)dx\wedge dy.$$

Proof. We prove the theorem only for ψ_1, here denoted for simplicity by ψ, the case of ψ_2 being similar. The symmetric space $SL(2,\mathbb{C})/SU(2)$ can be realized as an open set inside $SL(2,C)$ as the space of Hermitian positive definite 2×2 matrices of determinant 1. It is of constant curvature -1, equipped with the following metric: for any two $X, Y \in T_A SL(2,\mathbb{C})/SU(2) = \{\text{Hermitian trace free } 2\times 2 \text{ matrices}\}$,

$$\langle X, Y\rangle_A := \tfrac{1}{2}\mathrm{tr}(A^{-1}XA^{-1}Y).$$

Now we explain the relation between the Hermitian metric and the harmonic map more carefully. Select a positively oriented unitary frame $\{s_1, s_2\}$ of lifted bundle E over the base point $x \in \widetilde{\Sigma}$ which is fixed under the real structure $\tau_1(x)$. Parallel translate of the frame using the flat connection gives a global section of the unitary frame bundle, also denoted $\{s_1, s_2\}$, which is fixed under the real structure τ_1. The Hermitian metric on E gives a ρ-equivariant harmonic map as follows:

$$\psi : \widetilde{\Sigma} \to SL(2,\mathbb{C})/SU(2), \qquad y \longmapsto \{(H(s_i, s_j))\}_{i,j=1,2}.$$

As in the previous section, we also have a local holomorphic frame $\{e, e^*\}$. Denote the local frame change M such that $(s_1, s_2) = M(e, e^*)$. The Hermitian metric H and the harmonic map ψ are then locally related as $\psi = \bar{M}^t H M = \bar{M}^t \left(\begin{smallmatrix} k^{-1} & 0 \\ 0 & k\end{smallmatrix}\right) M$. And the Higgs field and the differential of ψ are related by (see page 375 in [4]), $-\frac{1}{2}\psi(z)^{-1}d\psi = M^{-1}(\phi dz + \phi^* d\bar{z})M$. By reality, the image of ψ lies in a subset of $SL(2,\mathbb{C})/SU(2)$ as an embedded hyperbolic plane $\mathbb{H}$, denoted as N,

$$N = \left\{ \begin{pmatrix} u & v \\ \bar{v} & u \end{pmatrix}, \text{where } u, w \in \mathbb{R}, v \in \mathbb{C}, u^2 - |v|^2 = 1 \right\}.$$

Hence the pull-back metric of ψ is

$$\begin{aligned} g_1 &= \langle \psi_z, \psi_z \rangle dz^2 + 2 \langle \psi_z, \psi_{\bar z} \rangle dz d\bar z + \langle \psi_{\bar z}, \psi_{\bar z} \rangle d\bar z^2 \\ &= 2\,\mathrm{tr}(\phi_1\phi_1) dz^2 + 4\,\mathrm{tr}(\phi_1\phi_1^*) dz d\bar z + 2\,\mathrm{tr}(\phi_1^*\phi_1^*) d\bar z^2, \end{aligned}$$

Part (1) now follows from a direct computation.

For part (2), at a point $A \in N$, there exists a $g \in SL(2,\mathbb{C})$ such that $A = \bar g^t g$, and the invariant volume form $\mathrm{Vol}(g_{\mathbb{H}})$ is $\sigma_1^* \wedge \sigma_2^*$, where

$$\left\{ \sigma_1 = \bar g^t \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) g, \quad \sigma_2 = \bar g^t \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right) g \right\},$$

is a positively oriented orthonormal frame. Note that the volume form does not depend on the choice of g.

From the expression $\psi(z) = \bar M^t \left(\begin{smallmatrix} k^{-1} & 0 \\ 0 & k \end{smallmatrix}\right) M$, one can write $\psi(z) = \bar g^t g$, where $g = \begin{pmatrix} k^{-\frac12} & 0 \\ 0 & k^{\frac12} \end{pmatrix} M$, and hence $\left\{ \sigma_1 = \bar M^t \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) M, \sigma_2 = \bar M^t \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right) M \right\}$ is a positively oriented orthonormal basis at $\psi(z)$. Denote

$$\begin{aligned} u_1 dx + u_2 dy = \phi_1 dz + \phi_1^* d\bar z &= \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right) dz + \begin{pmatrix} 0 & k^2\bar\beta \\ k^{-2}\bar\alpha & 0 \end{pmatrix} d\bar z \\ &= \begin{pmatrix} 0 & \alpha + k^2\bar\beta \\ \beta + k^{-2}\bar\alpha & 0 \end{pmatrix} dx + i \begin{pmatrix} 0 & \alpha - k^2\bar\beta \\ \beta - k^{-2}\bar\alpha & 0 \end{pmatrix} dy; \end{aligned}$$

then $-\frac12 \psi_x = \psi(z)(-\frac12 \psi(z)^{-1}\psi_x) = \bar M^t H u_1 M$, similarly, $-\frac12\psi_y = \bar M^t H u_2 M$. Moreover, we have $Hu_1 = \begin{pmatrix} 0 & k^{-1}\alpha + k\bar\beta \\ k\beta + k^{-1}\bar\alpha & 0 \end{pmatrix}, Hu_2 = i \begin{pmatrix} 0 & k^{-1}\alpha - k\bar\beta \\ k\beta - k^{-1}\bar\alpha & 0 \end{pmatrix}$.

Let $k^{-1}\alpha + k\bar\beta = a + bi, i(k^{-1}\alpha - k\bar\beta) = c + di$. Therefore $-\frac12\psi_x = a\sigma_1 + b\sigma_2, -\frac12\psi_y = c\sigma_1 + d\sigma_2$, and then the pull-back volume form is

$$\begin{aligned} \psi^*\mathrm{Vol}(g_{\mathbb{H}}) &= \mathrm{Vol}(g_{\mathbb{H}})(\psi_x, \psi_y) dx \wedge dy = \sigma_1^* \wedge \sigma_2^* \left(2a\sigma_1 + 2b\sigma_2, 2c\sigma_1 + 2d\sigma_2\right) dx \wedge dy \\ &= (4ad - 4bc) dx \wedge dy = 4\Re((k^{-1}\alpha + k\bar\beta)(k^{-1}\bar\alpha - k\beta)) dx \wedge dy \\ &= 4(k^{-2}|\alpha|^2 - k^2|\beta|^2) dx \wedge dy. \end{aligned}$$

□

5. Construction of AdS 3-manifolds

We say that the metric g_1 *strictly dominates* the metric g_2 if $g_1 - g_2$ is positive definite. In this case let us write $g_1 > g_2$. This condition depends on ρ_1, ρ_2, and on the choice of Σ. In this section we will prove the following theorem.

Theorem 5.1. *Let S be a closed surface. Given two representations ρ_1, ρ_2 and a conformal structure Σ such that $g_1 > g_2$, there exists an AdS structure with holonomy $\overline{\rho_1 \otimes \rho_2}$ on a circle bundle U over S with Euler class $|e_1 - e_2|$.*

With Theorem 5.1 we can construct AdS structures on all the circle bundles with even Euler class between 2 and $2g-2$. Our proof of the theorem gives a way to express the AdS metric in terms of coordinates on the manifold U depending on the solutions of Hitchin's equation. Using this we can easily see some properties of these structures, as we will see in the next sections.

Proof. We use the circle bundle U defined in the end of Section 3 and consider the pull-back bundle $\mathbb{M}_{\bar\rho} \to U$. Next we analyze the tautological section $s : U \to \mathbb{M}_{\bar\rho}$ following the plan sketched in Section 2. We will prove that given two representations ρ_1, ρ_2 and a complex structure Σ such that $g_1 > g_2$, the section s is a transverse section. This proves that there exists an AdS structure with holonomy $\overline{\rho_1 \otimes \rho_2}$.

We need to write the equations of U as a subset of E. Let $v \in U$. We have

$$v = \begin{pmatrix} 0 \\ \nu \\ \omega \\ 0 \end{pmatrix}, \qquad \tau v = \begin{pmatrix} 0 & 0 & 0 & hk \\ 0 & 0 & h^{-1}k & 0 \\ 0 & hk^{-1} & 0 & 0 \\ h^{-1}k^{-1} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \bar{\nu} \\ \bar{\omega} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ h^{-1}k\bar{\omega} \\ hk^{-1}\bar{\nu} \\ 0 \end{pmatrix}.$$

By reality, we have that $v = \tau v$, therefore $\omega = hk^{-1}\bar{\nu}$. Since $Q(v, v) = 1$, we have that $\nu\omega = 1$. Hence $|\nu| = (hk^{-1})^{-\frac{1}{2}}$.

We give a local coordinate description of the tautological section s. We can cover Σ with little open sets V_i bi-holomorphic to the unit disc $\{|z| < 1\}$. Locally over every V_i, U is $\{|z| < 1\} \times S^1$. Denote the function $g = (hk^{-1})^{-\frac{1}{2}}$. Given coordinates (x, y, θ), where $z = x + iy$, we write the section s and its derivatives at v:

$$s(v) = \begin{pmatrix} 0 \\ ge^{i\theta} \\ g^{-1}e^{-i\theta} \\ 0 \end{pmatrix}, \qquad \nabla_{\frac{\partial}{\partial\theta}} s = \begin{pmatrix} 0 \\ ige^{i\theta} \\ -ig^{-1}e^{-i\theta} \\ 0 \end{pmatrix},$$

$$\nabla_{\frac{\partial}{\partial z}} s = \begin{pmatrix} X \\ 0 \\ 0 \\ Y \end{pmatrix} + c\nabla_{\frac{\partial}{\partial\theta}} s, \qquad \nabla_{\frac{\partial}{\partial\bar{z}}} s = \begin{pmatrix} Z \\ 0 \\ 0 \\ W \end{pmatrix} + \bar{c}\nabla_{\frac{\partial}{\partial\theta}} s,$$

where $c = ig^{-1}\partial g$, and

$$X = \gamma ge^{i\theta} + \alpha g^{-1}e^{-i\theta}, \qquad Y = \beta ge^{i\theta} + \delta g^{-1}e^{-i\theta},$$
$$Z = h^2\bar{\delta}ge^{i\theta} + k^2\bar{\beta}g^{-1}e^{-i\theta}, \qquad W = k^{-2}\bar{\alpha}ge^{i\theta} + h^{-2}\bar{\gamma}g^{-1}e^{-i\theta}.$$

The transversality condition for the section s is equivalent to the condition that at every point $(x, y, \theta) \in U$, the derivatives $\nabla_{\frac{\partial}{\partial x}} s, \nabla_{\frac{\partial}{\partial y}} s, \nabla_{\frac{\partial}{\partial\theta}} s$ form a basis of the tangent space to the fiber of $\mathbb{M}_{\bar{\rho}}$ at $s(x, y, \theta)$, or equivalently, the four vectors $s, \nabla_{\frac{\partial}{\partial x}} s, \nabla_{\frac{\partial}{\partial y}} s, \nabla_{\frac{\partial}{\partial\theta}} s$ are linearly independent.

Suppose that there exists $a, b, c, d \in \mathbb{R}$ such that

$$a\nabla_{\frac{\partial}{\partial x}} s + b\nabla_{\frac{\partial}{\partial y}} s + c\nabla_{\frac{\partial}{\partial\theta}} s + ds = 0.$$

Written $\nabla_{\frac{\partial}{\partial z}} s, \nabla_{\frac{\partial}{\partial\bar{z}}} s$ instead of $\nabla_{\frac{\partial}{\partial x}} s, \nabla_{\frac{\partial}{\partial y}} s$, we have

$$(a + bi)\nabla_{\frac{\partial}{\partial z}} s + (a - bi)\nabla_{\frac{\partial}{\partial\bar{z}}} s + c\nabla_{\frac{\partial}{\partial\theta}} s + ds = 0.$$

Taking the first entry of the vector of $(a+bi)\nabla_{\frac{\partial}{\partial z}} s + (a-bi)\nabla_{\frac{\partial}{\partial\bar{z}}} s + c\nabla_{\frac{\partial}{\partial\theta}} s + ds$,

$$(a + bi)X + (a - bi)Z = 0. \tag{5.1}$$

If the metrics satisfy $g_1 > g_2$, by Lemma 5.2 proven below, we obtain that $a = b = 0$.

Therefore $c\nabla_{\frac{\partial}{\partial\theta}} s + ds = 0$, which clearly implies that $c = d = 0$. □

Lemma 5.2. *Suppose that the pull-back metrics satisfy $g_1 > g_2$. Then, if $u \in \mathbb{C}$ satisfies $uX + \bar{u}Z = 0$, we have $u = 0$.*

Proof. We carry out the following direct computation:

$$\begin{aligned} &uX + \bar{u}Z = 0 \Rightarrow u(\gamma ge^{i\theta} + \alpha g^{-1}e^{-i\theta}) + \bar{u}(h^2\bar{\delta}ge^{i\theta} + k^2\bar{\beta}g^{-1}e^{-i\theta}) = 0 \\ &\Rightarrow |u\gamma ge^{i\theta} + \bar{u}h^2\bar{\delta}ge^{i\theta}|^2 = |u\alpha g^{-1}e^{-i\theta} + \bar{u}k^2\bar{\beta}g^{-1}e^{-i\theta}|^2 \\ &\Rightarrow |uh^{-1}\gamma + \bar{u}h\bar{\delta}|^2 = |uk^{-1}\alpha + \bar{u}k\bar{\beta}|^2 \quad \text{since } \ g = (hk^{-1})^{-\frac{1}{2}} \\ &\Rightarrow \gamma\delta u^2 + |u|^2(h^{-2}|\gamma|^2 + h^2|\delta|^2) + \bar{\delta}\bar{\gamma}\bar{u}^2 = \alpha\beta u^2 + |u|^2(k^{-2}|\alpha|^2 + k^2|\beta|^2) + \bar{\alpha}\bar{\beta}\bar{u}^2. \end{aligned}$$

Applying Theorem 4.1

$$\Rightarrow g_1\left(u\frac{\partial}{\partial z},\bar{u}\frac{\partial}{\partial \bar{z}}\right) = g_2\left(u\frac{\partial}{\partial z},\bar{u}\frac{\partial}{\partial \bar{z}}\right) \Longrightarrow u = 0, \quad \text{since} \quad g_1 > g_2. \qquad \square$$

6. Comparison of domination conditions

In this section we will discuss some consequences of the condition of domination between pull-back metrics we introduced in the previous section and how this condition is related to the condition of domination defined by Salein. In this way we see how our Theorem 5.1 gives a new proof of Salein's theorem in [18].

Proposition 6.1. *Let ρ_1, ρ_2 be two representations such that $g_1 > g_2$. Then ρ_1 is Fuchsian, $|e_1| = 2g-2$, the harmonic map ψ_1 is a diffeomorphism, g_1 is a hyperbolic metric, ρ_2 is not Fuchsian, and $|e_2| < 2g-2$.*

Proof. If $g_1 > g_2$, then g_1 is positive definite. Hence ψ_1 is a local diffeomorphism. Since ψ_1 is also ρ_1-equivariant, it is the developing map of a hyperbolic structure on S with holonomy ρ_1. Hence ρ_1 is a Fuchsian representation, and, by [8], $|e_1| = 2g-2$. As the developing map of a hyperbolic structure, ψ_1 is a diffeomorphism.

If ρ_2 were also Fuchsian, the condition $g_1 > g_2$ would give two hyperbolic metrics on the same surface with different areas, which is impossible, hence $|e_2| < 2g-2$. $\square$

This condition of strict domination between the pull-back metrics is very related to another notion of domination between the two representations ρ_1, ρ_2: we say that ρ_1 *strictly dominates* ρ_2 if there exists a (ρ_1, ρ_2)-equivariant map $f : \mathbb{H} \to \mathbb{H}$ with Lipschitz constant strictly smaller than 1. The latter notion was introduced in [18] and it only depends on the two representations ρ_1, ρ_2. The following proposition comes from Tholozan [21] and Deroin-Tholozan [5].

Proposition 6.2. *Given two representations ρ_1, ρ_2 in $SL(2,\mathbb{R})$, there exists a conformal structure Σ such that $g_1 > g_2$ if and only if ρ_1 strictly dominates ρ_2.*

Proof. If there exists a Σ such that $g_1 > g_2$, we have seen in the previous proposition that ψ_1 is invertible. The map $\psi_2 \circ \psi_1^{-1} : \mathbb{H} \to \mathbb{H}$ is (ρ_1, ρ_2)-equivariant and has Lipschitz constant strictly smaller than 1.

The converse is the interesting implication. It is proved in [21] that if ρ_1 strictly dominates ρ_2, there exists a unique holomorphic structure Σ such that the corresponding Higgs bundles have $\alpha\beta = \gamma\delta$. In this case, it is proved in [5] that the difference of the pull-back metrics is positive definite. $\square$

Combining Theorem 5.1 and Proposition 6.2, we obtain a new proof of the following theorem, originally proven by Salein in [18], and later reproven in [10], which is one of the main theorems in the theory of AdS 3-manifolds:

Theorem 6.3. *Let S be a closed surface. Given two representations ρ_1, ρ_2 of $\pi_1(S)$ in $SL(2,\mathbb{R})$ such that ρ_1 strictly dominates ρ_2, there exists an AdS structure with holonomy $\overline{\rho_1 \otimes \rho_2}$ on a circle bundle U over S with Euler class $|e_1 - e_2|$.*

7. Minimal immersions and fibers of the circle bundle

Using the above construction, we can understand how the circle fibers of U are related to the AdS structure. Let $z \in \Sigma$, and let C be the fiber of z in U, topologically a circle. By fixing a base point $c \in C$, the circle C becomes a loop in U. Let $\widetilde{C}$ be a lift of this loop to the universal covering $\widetilde{U}$. We now describe the curve $D(\widetilde{C})$.

Theorem 7.1. *The developing image $D(\widetilde{C})$ is a time-like geodesic loop that turns once around $\mathbb{M}$.*

Proof. We can compute the tangent vector along the fiber and its derivative:

$$s = \begin{pmatrix} 0 \\ ge^{i\theta} \\ g^{-1}e^{-i\theta} \\ 0 \end{pmatrix}, \qquad \nabla_{\frac{\partial}{\partial\theta}} s = \begin{pmatrix} 0 \\ ige^{i\theta} \\ -ig^{-1}e^{-i\theta} \\ 0 \end{pmatrix}, \qquad \nabla_{\frac{\partial}{\partial\theta}} \nabla_{\frac{\partial}{\partial\theta}} s = - \begin{pmatrix} 0 \\ ge^{i\theta} \\ g^{-1}e^{-i\theta} \\ 0 \end{pmatrix} = -s.$$

From the second formula $Q(\nabla_{\frac{\partial}{\partial\theta}} s, \nabla_{\frac{\partial}{\partial\theta}} s) = 1 > 0$, hence the fiber is time-like, from the third formula, it is a geodesic. The flat connection on p^*E is the pull back of a flat connection on E, hence it is trivial on the fiber. So the explicit formula for s shows that $D(\widetilde{C})$ is a loop and it turns once around $\mathbb{M}$. □

Remark 7.2. Gueritaud and Kassel in [10] also found a parametrization of AdS manifolds as circle bundles such that the circle fibers develop to time-like geodesics. It is not clear if our parametrization is the same as theirs.

We can use the structure of the circle fibers of our AdS structure on U to construct and characterize minimal immersions of Riemann surfaces into quadrics. This is a new result of this paper. We remark that this construction is not restricted to the case where the section is transverse.

The group $O(2,2)$ acts on $\mathrm{Gr}(2,4)$, the Grassmannian of 2-planes in $\mathbb{R}^4$, preserving the open subset $\mathrm{Gr}^+(2,4)$, the Grassmannian of time-like 2-planes in $(\mathbb{R}^4, Q)$. The bundle E over Σ has structure group $O(2,2)$, by changing fiber we get the bundle $E(\mathrm{Gr}^+(2,4))$ with fiber $\mathrm{Gr}^+(2,4)$ and the same structure group. This bundle inherits also a flat structure. Every circle fiber of U corresponds to a time-like geodesic, i.e., to a time-like 2-plane in the corresponding fiber of E. This gives a section of $E(\mathrm{Gr}^+(2,4))$. As in Section 2, it induces a $\rho_1 \otimes \rho_2$-equivariant map

$$f : \widetilde{\Sigma} \to \mathrm{Gr}^+(2,4).$$

We are going to show that f is harmonic and that, under some hypothesis, it is also a minimal immersion. Before we state the exact result, we need to introduce a pseudo-Riemannian metric on $\mathrm{Gr}^+(2,4)$.

The *Plücker embedding* identifies $\mathrm{Gr}^+(2,4)$ with a subset of a projective space:

$$\mathrm{Gr}^+(2,4) \ni \mathrm{Span}(v,w) \to [v \wedge w] \in \mathbb{P}\left(\Lambda^2\mathbb{R}^4\right).$$

This projective embedding can be lifted to an embedding in $\Lambda^2\mathbb{R}^4$. Fix two independent space-like vectors w_1, w_2 and an orientation of $\mathbb{R}^4$. The lifted embedding is given by

$$\mathrm{Gr}^+(2,4) \ni \mathrm{Span}(v_1,v_2) \to v_1 \wedge v_2 \in \Lambda^2\mathbb{R}^4,$$

where v_1, v_2 are Q-orthonormal and v_1, v_2, w_1, w_2 are a positively oriented basis of $\mathbb{R}^4$. The element $v_1 \wedge v_2$ does not depend on the choice of a Q-orthonormal basis.

The wedge product induces a symmetric bilinear form $v \wedge w$ on $\Lambda^2\mathbb{R}^4$ of signature $(3,3)$. This restricts to a signature $(2,2)$ pseudo-Riemannian metric on the submanifold $\mathrm{Gr}^+(2,4)$ that does not depend on the choice of w_1, w_2 and the orientation of $\mathbb{R}^4$, hence it is preserved by the action of $O(2,2)$.

We remark that, by a quite general result about real projective structures (see [1, Lemma 3.5.0.1]) the image of the differential of f lies in a definite subspace of the tangent space to $\mathrm{Gr}^+(2,4)$ if and only if the section is transverse.

Now let us prove that the map f is harmonic. Locally, the map f can be written as $f = s \wedge \nabla_{\frac{\partial}{\partial\theta}} s$. For simplicity, we will denote $\nabla_{\frac{\partial}{\partial\theta}} s, \nabla_{\frac{\partial}{\partial z}} s, \nabla_{\frac{\partial}{\partial \bar z}} s$ by $s_\theta, s_z, s_{\overline{z}}$.

Theorem 7.3. *The map f is harmonic, and it is the unique $\rho_1 \otimes \rho_2$-equivariant harmonic map from $\widetilde{\Sigma}$ to $Gr^+(2,4)$.*

Proof. Since the domain of the map f is 2-dimensional, the condition that f is harmonic is equivalent to $\widetilde{\nabla}_{\bar z} f_z = 0$, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\mathrm{Gr}^+(2,4)$ (see [11] page 425). We can compute f_z:

$$f_z = (s \wedge s_\theta)_z = s_z \wedge s_\theta + s \wedge s_{\theta,z}.$$

Differentiating s_z with respect to θ, $s_{z,\theta} = \begin{pmatrix} i\gamma g e^{i\theta} - i\alpha g^{-1} e^{-i\theta} \\ 0 \\ 0 \\ i\beta g e^{i\theta} - i\delta g^{-1} e^{-i\theta} \end{pmatrix} - cs$, using $\nabla_{\frac{\partial}{\partial\theta}} \nabla_{\frac{\partial}{\partial\theta}} s = -s$.

In the coordinates given by the six minors of a 2×4 matrix, we have

$$s_z \wedge s_\theta = (i\gamma g^2 e^{2i\theta} + i\alpha, -i\gamma - i\alpha g^{-2} e^{-2i\theta}, 0, 0, -i\beta g^2 e^{2i\theta} - i\delta, i\beta + i\delta g^{-2} e^{-2i\theta}),$$
$$s \wedge s_{\theta,z} = (-i\gamma g^2 e^{2i\theta} + i\alpha, -i\gamma + i\alpha g^{-2} e^{-2i\theta}, 0, 0, i\beta g^2 e^{2i\theta} - i\delta, i\beta - i\delta g^{-2} e^{-2i\theta}),$$

$f_z = (2i\alpha, -2i\gamma, 0, 0, -2i\delta, 2i\beta)$, and since it is holomorphic, $\widetilde{\nabla}_{\bar z} f_z = 0$.

For uniqueness, note that $\mathrm{Gr}^+(2,4)$ is isometric to $(\mathbb{H}^2, h) \times (\mathbb{H}^2, -h)$ (see Torralbo [25]). The harmonicity of a map into a product is equivalent to the harmonicity on each factor. By Donaldson [6], the ρ_i-equivariant harmonic maps into $(\mathbb{H}^2, h)$ are unique. Hence the equivariant harmonic map into $\mathrm{Gr}^+(2,4)$ is unique. ☐

In the next theorem we will prove that the map f is a minimal immersion if and only if Σ is the conformal structure with vanishing Pfaffian (see Section 4). This gives a geometric interpretation of the conformal structure with vanishing Pfaffian and it gives a link between equivariant minimal immersions and AdS structures. This relationship between conformality and the vanishing Pfaffian is in contrast with the case for $Sp(4,\mathbb{R})$, studied by Baraglia [1], where the conformality comes from the vanishing of the Hopf differential.

Theorem 7.4. *The harmonic map f is conformal if and only if $\alpha\beta = \gamma\delta$. In this case f is a $\rho_1 \otimes \rho_2$-equivariant minimal immersion into $Gr^+(2,4)$. Moreover, there exists a $\rho_1 \otimes \rho_2$-equivariant minimal immersion $f : \widetilde{S} \to Gr^+(2,4)$ if and only if ρ_1 strictly dominates ρ_2. In this case, the minimal immersion is unique.*

Proof. The condition that the map f is conformal is equivalent to $\langle f_z, f_z \rangle = 0$, where the pairing is the pseudo-Riemannian structure on $\mathrm{Gr}^+(2,4)$,

$$\langle f_z, f_z \rangle = f_z \wedge f_z = 2 s_z \wedge s_\theta \wedge s \wedge s_{\theta,z}$$
$$= 2 \det \begin{pmatrix} \gamma g e^{i\theta} + \alpha g^{-1} e^{-i\theta} & 0 & 0 & i\gamma g e^{i\theta} - i\alpha g^{-1} e^{-i\theta} \\ 0 & i g e^{i\theta} & g e^{i\theta} & 0 \\ 0 & -i g^{-1} e^{-i\theta} & g^{-1} e^{-i\theta} & 0 \\ \beta g e^{i\theta} + \delta g^{-1} e^{-i\theta} & 0 & 0 & i\beta g e^{i\theta} - i\delta g^{-1} e^{-i\theta} \end{pmatrix} d\mathrm{Vol}_{\mathbb{R}^4}$$
$$= -8(\alpha\beta - \gamma\delta) d\mathrm{Vol}_{\mathbb{R}^4}.$$

Hence f is conformal if and only if $\alpha\beta = \gamma\delta$. This also gives the "if" part of the second statement.

To see the "only if" part, given a $\rho_1 \otimes \rho_2$-equivariant minimal immersion u into $\mathrm{Gr}^+(2,4)$, let Σ be the pull-back conformal structure on the surface. Then u is an equivariant harmonic map from $\widetilde{\Sigma}$ into $\mathrm{Gr}^+(2,4)$. By the uniqueness of such harmonic maps, u agrees with the f constructed above. This f is conformal, hence $\alpha\beta = \delta\gamma$. By the result of Deroin-Tholozan [5], ρ_1 strictly dominates ρ_2. The uniqueness of minimal immersions follows from the uniqueness of the conformal structure with vanishing Pfaffian for a given representation by Tholozan [21]. $\square$

8. Volume of AdS 3-manifolds

The computation of the volume of closed AdS 3-manifolds was Question 2.3 in the list of open problems [2]. It was first answered some months before us in Tholozan's thesis [22]; see also [23]. Labourie [17] then related the volumes of AdS 3-manifolds with the Chern-Simons invariants. Here we give a different proof of the formula for the volume, based on our construction of the AdS structures.

Theorem 8.1. *The volume of* $\mathbb{M}/\rho_1 \otimes \rho_2$ *is* $\pi^2|e_1 + e_2|$.

Note here $|e_1| = 2g-2$. This formula differs from the one in [22, Lemma 4.5.2] by a factor 2 because the model space used there is a quotient of $\mathbb{M}$ by $\mathbb{Z}/2\mathbb{Z}$.

Proof. The quotient $\mathbb{M}/\rho_1 \otimes \rho_2$ is the AdS structure we constructed on U in Section 5. Denote G as the matrix presentation of the Lorentzian metric tensor in terms of the basis $s_z, s_{\bar z}, s_\theta$. The volume form is $d\mathrm{Vol} = |\sqrt{|\det G|}dz \wedge d\bar z \wedge d\theta|$.

By direct calculation, we see that $Q(s_\theta, s_\theta) = 1$, $Q(s_z, s_\theta) = c$, $Q(s_{\bar z}, s_\theta) = \bar c$, $Q(s_z, s_z) = -XY + c^2$, $Q(s_{\bar z}, s_{\bar z}) = -ZW + \bar c^2$, $Q(s_z, s_{\bar z}) = -\frac{1}{2}(XW + YZ) + c\bar c$.

Hence $G = \begin{pmatrix} -XY + c^2 & -\frac{1}{2}(XW+YZ) + c\bar c & c \\ -\frac{1}{2}(XW+YZ) + c\bar c & -ZW + \bar c^2 & \bar c \\ c & \bar c & 1 \end{pmatrix}$, and then

$$\det(G) = XY \cdot ZW - \frac{1}{4}(XW+YZ)^2 = -\frac{1}{4}(XW - YZ)^2.$$

Therefore the volume of an AdS 3-manifold is

$$\int_U d\mathrm{Vol} = \frac{1}{2}\int_U |XW - YZ| dz \wedge d\bar z \wedge d\theta.$$

The non-degeneracy of volume form implies $XW - YZ$ has constant sign on U,

$$\begin{aligned} &= \frac{1}{2}\left|\int_U (XW - YZ) dz \wedge d\bar z \wedge d\theta\right| \\ &= \frac{1}{2}\left|\int_\Sigma \int_{S^1} ((\gamma\bar\alpha g^2 e^{2i\theta} k^{-2} + |\alpha|^2 k^{-2} + |\gamma|^2 h^{-2} + \bar\gamma\alpha h^{-2} g^{-2} e^{-2i\theta}) \right. \\ &\qquad \left. - (\beta\bar\delta g^2 e^{2i\theta} h^2 + |\delta|^2 h^2 + |\beta|^2 k^2 + \delta\bar\beta k^2 g^{-2} e^{-2i\theta})) dz \wedge d\bar z \wedge d\theta \right| \\ &= 2\pi \left| \int_\Sigma ((|\alpha|^2 k^{-2} - |\beta|^2 k^2) + (|\gamma|^2 h^{-2} - |\delta|^2 h^2)) dx \wedge dy \right|. \end{aligned}$$

Applying Theorem 4.1 and equation (4.1),

$$= \frac{1}{2}\pi \left| \int_\Sigma \mathrm{Vol}(g_1) + \mathrm{Vol}(g_2) \right| = \pi^2 |e_1 + e_2|.$$

$\square$

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References

[1] D. Baraglia, *G_2 geometry and integrable systems*, Ph.D. Thesis, 2009, arXiv:1002.1767.
[2] T. Barbot, F. Bonsante, J. Danciger, W. Goldman, F. Guéritaud, F. Kassel, K. Krasnov, J.-M. Schlenker, and A. Zeghib, *Some open questions on Anti-de Sitter geometry*, arXiv:1205.6103.
[3] Marc Burger, Alessandra Iozzi, and Anna Wienhard, *Higher Teichmüller spaces: from* $\mathrm{SL}(2,\mathbb{R})$ *to other Lie groups*, Handbook of Teichmüller theory. Vol. IV, IRMA Lect. Math. Theor. Phys., vol. 19, Eur. Math. Soc., Zürich, 2014, pp. 539–618, DOI 10.4171/117-1/14. MR3289711
[4] Kevin Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361–382. MR965220
[5] Bertrand Deroin and Nicolas Tholozan, *Dominating surface group representations by Fuchsian ones*, Int. Math. Res. Not. IMRN **13** (2016), 4145–4166, DOI 10.1093/imrn/rnv275. MR3544632
[6] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 127–131, DOI 10.1112/plms/s3-55.1.127. MR887285
[7] W. Goldman, *Geometric structures and varieties of representations*, preprint.
[8] William M. Goldman, *Topological components of spaces of representations*, Invent. Math. **93** (1988), no. 3, 557–607, DOI 10.1007/BF01410200. MR952283
[9] P. Gothen, *The topology of the Higgs bundle moduli space*, Ph.D. Thesis, 1995.
[10] F. Gueritaud and F. Kassel, *Maximally stretched laminations on geometrically finite hyperbolic manifolds*, to appear in Geom. Topol.
[11] Frédéric Hélein and John C. Wood, *Harmonic maps*, Handbook of global analysis, Elsevier Sci. B. V., Amsterdam, 2008, pp. 417–491, 1213, DOI 10.1016/B978-044452833-9.50009-7. MR2389639
[12] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 59–126, DOI 10.1112/plms/s3-55.1.59. MR887284
[13] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473, DOI 10.1016/0040-9383(92)90044-I. MR1174252
[14] F. Kassel, *Quotients compacts d'espaces homogènes réels ou p-adiques*, Ph.D. Thesis, 2009.
[15] Bruno Klingler, *Complétude des variétés lorentziennes à courbure constante* (French), Math. Ann. **306** (1996), no. 2, 353–370, DOI 10.1007/BF01445255. MR1411352
[16] Ravi S. Kulkarni and Frank Raymond, *3-dimensional Lorentz space-forms and Seifert fiber spaces*, J. Differential Geom. **21** (1985), no. 2, 231–268. MR816671
[17] F. Labourie, *Chern-Simons invariant and Tholozan volume formula*, Seminar talk at MSRI, https://www.msri.org/seminars/21432.
[18] François Salein, *Variétés anti-de Sitter de dimension 3 exotiques* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **50** (2000), no. 1, 257–284. MR1762345
[19] Carlos T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), no. 4, 867–918, DOI 10.2307/1990994. MR944577
[20] Carlos T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95. MR1179076
[21] N. Tholozan, *Dominating surface group representations and deforming closed Anti-de Sitter 3-manifolds*, to appear in Geom. Topol.
[22] N. Tholozan, *Uniformisation des variétés pseudo-riemanniennes localement homogènes*, Ph.D. Thesis, 2014.
[23] N. Tholozan, *The volumes of complete Anti-de Sitter* 3*-manifolds*, arXiv:1509.04178.
[24] William P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy. MR1435975

[25] Francisco Torralbo, *Minimal Lagrangian immersions in* $\mathbb{RH}^2 \times \mathbb{RH}^2$, Symposium on the Differential Geometry of Submanifolds, [s.n.], [s.l.], 2007, pp. 217–219. MR2509502

[26] Michael Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), no. 2, 449–479. MR982185

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